# Tensor Representations of the General Linear Super Group

R. Fioresi [1]

*Dipartimento di Matematica, Università di Bologna*
*Piazza di Porta S. Donato, 5. 40127 Bologna. Italy.*
e-mail: fioresi@dm.unibo.it

**Abstract**

We show a correspondence between tensor representations of the super general linear group $GL(m|n)$ and tensor representations of the general linear superalgebra $\mathfrak{gl}(m|n)$ constructed by Berele and Regev in [3].

## 1 Introduction

Supersymmetry is an important mathematical tool in physics that enables to treat on equal grounds the two types of elementary particles: bosons and fermions, whose states are described respectively by commuting and anti-commuting functions. It is fundamental to seek a unified treatment for these particles since they do transform into each other. Hence a symmetry that keeps one type separated from the other is not acceptable. For this reason the symmetries of elementary particles must be described not by groups, but by *supergroups*, which are a natural generalization of groups in the $\mathbf{Z}_2$ graded or *super* setting.

The theory of representations of supergroups has a particular importance since it is attached to the problem of the classification of elementary particles. For a more detailed historical and physical introduction to supersymmetry see the beautiful treatment in [17] 1.7, 1.8.

As in the classical theory, in order to understand the representations of a supergroup, one must first study the representations of its Lie superalgebra.

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The representation theory of the general linear superalgebra $\mathfrak{gl}(m|n)$ has been the object of study of many people.

In [3] Berele and Regeev provide a full account of a class of irreducible representations of $\mathfrak{gl}(m|n)$ that turns out to be linked to certain Young tableaux called *semistandard or superstandard tableax*. The same result appeares also in [6] by Dondi and Jarvis in a slightly different setting. Dondi and Jarvis in fact introduce the notion of *super permutation* and use this definition to motivate the semistandard Young tableaux used for the description of the irreducible representations of the general linear superalgebra.

The results by Berele and Regeev were later generalized and deepened by Brini, Regonati and Teolis in [4]. In their important work, they develop a unified theory that treats simoultaneosly the super and the classical case, through the powerful method of *virtual variables*.

Another account of this subject is found in [16]. Sergeev establishes a correspondence between a class of irreducible tensor representations of $\mathfrak{gl}(m|n)$ and the irreducible representations of a certain finite group, though different from the permutation group used both in [3] and [6].

It is important to remark at this point that the theory of representations of superalgebras and of supergroups has dramatic differences with respect to the classical theory. As we will see, not all representations of the super general linear group and its Lie superalgebra are found as tensor representations. Moreover not all representations are completely reducible over $\mathbf{C}$.

In this paper we want to understand how representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ can be naturally associated to the representation of the corresponding group $GL(m|n)$. Though this fact is stated in the physicists works as for example [6], [2], it is never satisfactorily worked out. Using [3, 6] we are able then to obtain a full classification of the irreducible tensor representations of the general linear supergroup coming from the natural diagonal action. We will do this using the approach suggested by Deligne and Morgan in [5]: using *the functor of points*.

In fact while in non commutative geometry in general the geometric object is lost and the only informations are retrieved through various algebras, like the $C^\infty$ functions or the algebraic functions, naturally associated to it; in supergeometry, using the functor of points approach, one is able to recover the geometric intuition, which otherwise would be lost.

This paper is organized as follows.

In section 2 we review some of the basic definitions of supergeometry. Since we will adopt the functorial language we relate our definitions to the other definitions appearing in the literature.

In section 3 we recall briefly the results obtained indipendently by Berele, Regev and Dondi, Jarvis. These results establish a correspondence between tensor representations of the permutation group and tensor representations of the superalgebra $\mathfrak{gl}(m|n)$. Moreover we show that the tensor representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ do not exhaust all polynomial representations of $\mathfrak{gl}(m|n)$.

Finally in section 4 we discuss tensor representations of the general linear supergroup associated to the representations of $\mathfrak{gl}(m|n)$ described in §3.

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## 2 Basic definitions

Let $k$ be an algebraically closed field of characteristic 0.

All algebras have to be intended over $k$.

A *superalgebra* $A$ is a $\mathbf{Z}_2$-graded algebra, $A = A_0 \oplus A_1$, $p(x)$ will denote the parity of an homogeneous element $x$. $A$ is said to be *commutative* if

$$xy = (-1)^{p(x)p(y)}yx$$

and its category will be denoted by (salg).

The concept of an affine supervariety or more generally an affine superscheme can be defined very effectively through its functor of points.

**Definition 2.1.** An *affine superscheme* is a representable functor:

$$\begin{array}{cccc} \mathbf{X}: & (\text{salg}) & \longrightarrow & (\text{sets}) \\ & A & \mapsto & X(A) = \text{Hom}(k[X], A) \end{array}$$

From this definition one can see that the category (salg) plays a role in algebraic supergeometry similar to the category of commutative algebras for the ordinary (i.e. non super) algebraic geometry. In particular it is possible to show that there is an equivalence of categories between the categories of affine superschemes and commutative superalgebras. (For more details see [7]).

**Examples 2.2.** 1. *Affine superspace.* Let $V = V_0 \oplus V_1$ be a finite dimensional super vector space. Define the following functor:

$$\mathbf{V} : (\text{salg}) \longrightarrow (\text{sets}), \qquad \mathbf{V}(A) = (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1$$

This functor is representable and it is represented by:

$$k[V] = Sym(V_0) \otimes \wedge(V_1)$$

where $Sym(V_0)$ is the polynomial algebra over the vector space $V_0$ and $\wedge(V_1)$ the exterior algebra over the vector space $V_1$. Let's see this more in detail.

If we choose a graded basis for $V$, $e_1 \dots e_m, \epsilon_1 \dots \epsilon_n$, with $e_i$ even and $\epsilon_j$ odd, then

$$k[V] = k[x_1 \dots x_m, \xi_1 \dots \xi_n],$$

where the latin letters denote commuting indeterminates, while the greek ones anticommuting indeterminates i.e. $\xi_i \xi_j = -\xi_j \xi_i$. In this case $V$ is commonly denoted with $k^{m|n}$ and $m|n$ is called the *superdimension* of $V$. We also will call $\mathbf{V}$ as the *functor of points* of the super vector space $V$.

Observe that:

$$\mathbf{V}(A) = \{(a_1 \dots a_m, \alpha_1 \dots \alpha_n) \quad | \quad a_i \in A_0, \quad \alpha_j \in A_1\} =$$

$$\text{Hom}(k[V], A) = \{\phi : k[V] \longrightarrow k \quad | \quad \phi(x_i) = a_i, \quad \phi(\xi_j) = \alpha_j\}$$

Hence $\mathbf{V}(A) = A_0 \otimes k^m \oplus A_1 \otimes k^n$.

2. *Tensor superspace.* We define the vector space of $r$-tensors as:

$$T^r(V) =_{\text{def}} \underbrace{V \otimes V \cdots \otimes V}_{r \text{ times}}$$

$T^r(V)$ is a super vector space, the parity of a monomial element is defined as $p(v_1 \otimes \cdots \otimes v_r) = p(v_1) + \cdots + p(v_r)$. $T^r(V)$ is also a supervariety functor:

$$\mathbf{T^r}(\mathbf{V})(A) = \mathbf{V}(A) \otimes_A \cdots \otimes_A \mathbf{V}(A)$$

We define the superspace of tensors $T(V)$ as:

$$T(V) = \bigoplus_{r \geq 0} T^r(V)$$

and denote with $\mathbf{T}(\mathbf{V})$ its functor of points.

3. *Supermatrices.* Given a finite dimensional super vector space $V$, the endomorphisms $\mathrm{End}(V)$ over $V$ is itself a supervector space: $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$, where $\mathrm{End}(V)_0$ are the endomorphisms preserving parity, while $\mathrm{End}(V)_1$ are those reversing parity.

Hence we can define the following functor:

$$\mathbf{End}(V) : (\text{salg}) \longrightarrow (\text{sets}), \qquad \mathbf{End}(V)(A) = (A \otimes \mathrm{End}(V))_0$$

This functor is representable (see (1)). Choosing a graded basis for $V$, $V = k^{m|n}$, the functor is represented by $k[x_{ij}, y_{il}, \xi_{kj}, \eta_{kl}]$ where $1 \leq i, j \leq m$, $m + 1 \leq k, l \leq m + n$.

In this case:

$$\mathbf{End}(V)(A) = \left\{ \begin{pmatrix} a_{m \times m} & \beta_{m \times n} \\ \gamma_{n \times m} & d_{n \times n} \end{pmatrix} \right\}$$

where $a$, $d$ and $\beta$, $\gamma$ are block matrices with respectively even and odd entries.

**Definition 2.3.** An *affine supergroup* $G$ is a group valued affine superscheme, i.e. it is a representable functor:

$$\begin{array}{rccc} \mathbf{G} : & (\text{salg}) & \longrightarrow & (\text{groups}) \\ & A & \mapsto & \mathbf{GL}(V)(A) \end{array}$$

It is simple to verify that the superalgebra representing the supergroup $\mathbf{G}$ has an Hopf superalgebra structure. More is true: Given a supervariety $\mathbf{G}$, $\mathbf{G}$ is a supergroup if and only if the algebra representing it $k[G]$ is an Hopf superalgebra.

Let $V$ be a finite dimensional super vector space. We are interested in the *general linear supergroup* $\mathbf{GL}(V)$.

**Definition 2.4.** We define *general linear supergroup* the functor

$$\begin{array}{cccc} \mathbf{GL}(V): & (\text{salg}) & \longrightarrow & (\text{sets}) \\ & A & \mapsto & \mathbf{GL}(V)(A) \end{array}$$

where $\mathbf{GL}(V)(A)$ is the set of automorphisms of the $A$-supermodule $A \otimes V$, $A \in$ (salg). More explicitly if $V = k^{m|n}$, the functor $\mathbf{GL}(V)$ commonly denoted $\mathbf{GL}(m|n)$ is defined as the set of automorphisms of $A^{m|n} =_{\text{def}} A \otimes k^{m|n}$ and is given by:

$$\mathbf{GL}(m|n)(A) = \left\{ \begin{pmatrix} a_{m\times m} & \beta_{m\times n} \\ \gamma_{n\times m} & d_{n\times n} \end{pmatrix} \quad | \quad a, d \quad \text{invertible} \right\}$$

where $a$, $d$ and $\beta$, $\gamma$ are block matrices with respectively even and odd entries.

This functor is representable and it is represented by the Hopf algebra (see [8]):

$$k[x_{ij}, y_{\alpha\beta}, \xi_{i\beta}, \eta_{\alpha j}, z, w]/\big((w \det(x) - 1, z \det(y) - 1\big),$$

$$i, j = 1, \dots m \qquad \alpha, \beta = 1, \dots n.$$

We now would like to introduce the notion of Lie superalgebra using the functorial language. We then see it is equivalent to the more standard definitions (see [11] for example).

**Definition 2.5.** Let $\mathfrak{g}$ be a finite dimensional supervector space. The functor (see Example 2.2 (1)):

$$\mathbf{g} : (\text{salg}) \longrightarrow (\text{sets}), \qquad \mathbf{g}(A) = (A \otimes \mathbf{g})_0$$

is said to be a *Lie superalgebra* if it is Lie algebra valued, i.e. for each $A$ there exists a linear map:

$$[\ ,\ ]_A : \mathbf{g}(A) \times \mathbf{g}(A) \longrightarrow \mathbf{g}(A)$$

satisfying the antisymmetric property and the Jacobi identity.

Notice that in the same way as the supergroup functor is group valued, the Lie superalgebra functor is Lie algebra valued, i. e. it has values in a *classical category*. The super nature of these functors arises from the different starting category, namely (salg), which allows superalgebras as representing objects.

The usual notion of Lie superalgebra, as defined for example by Kac in [11] is equivalent to this functorial definition. Let's recall this definition and see the equivalence with the Definition 2.5 more in detail.

**Definition 2.6.** Let $\mathfrak{g}$ be a super vector space. We say that a bilinear map

$$[,]:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$$

is a *superbracket* if $\forall x,y,z\in\mathfrak{g}$:

$$[x,y]=(-1)^{p(x)p(y)}[y,x]$$

$$[x,[y,z]]+(-1)^{p(x)p(y)+p(x)p(z)}[y,[z,x]]+(-1)^{p(x)p(z)+p(y)p(z)}[z,[x,y]]=0$$

$(\mathfrak{g},[,])$, is what in the literature is commonly defined as *Lie superalgebra*.

**Observation 2.7.** The two concepts of Lie superalgebra $\mathbf{g}$ in the functorial setting and superbracket on a supervector space $(\mathfrak{g},[,])$ are equivalent.

In fact if we have a Lie superalgebra $\mathbf{g}$ there is always a superspace $\mathfrak{g}$ associated to it together with a superbracket. The superbracket on $\mathfrak{g}$ is given following the *even rules*. (For a complete treatment of even rules see pg 57 [5]). Given $v,w\in\mathfrak{g}$, we have that since the Lie bracket on $\mathbf{g}(A)$ is $A$-linear:

$$[a\otimes v,b\otimes w]=ab\otimes z\in(A\otimes\mathfrak{g})_0=\mathfrak{g}(A)$$

Hence we can define the bracket $\{v,w\}$ as the element of $\mathfrak{g}$ such that: $z=(-1)^{p(a)p(w)}\{v,w\}$ i. e. satisfying the relation:

$$[a\otimes b,b\otimes w]=(-1)^{p(b)p(v)}ab\otimes\{v,w\}$$

We need to check it si a superbracket. Let's see for example the antisymmetry property. Observe first that if $a\otimes v\in(\mathfrak{g}\otimes A)_0$ must be $p(v)=p(a)$, since $(A\otimes\mathfrak{g})_0=A_0\otimes\mathfrak{g}_0\oplus A_1\otimes\mathfrak{g}_1$. So we can write:

$$[a\otimes v,b\otimes w]=(-1)^{p(b)p(v)}ab\otimes\{v,w\}=(-1)^{p(v)p(w)}ab\otimes\{v,w\}$$

On the other hand:

$$\begin{aligned}
[b\otimes w, a\otimes v] \quad &= (-1)^{p(a)p(w)} ba\otimes \{w,v\} = \\
&= (-1)^{p(a)p(w)+p(a)p(b)} ab\otimes \{w,v\} = \\
&= (-1)^{2p(w)p(v)} ab\otimes \{w,v\} = ab\otimes \{w,v\}.
\end{aligned}$$

Comparing the two expression we get the antisymmetry of the superbracket. For the super Jacobi identity the calculation is the same.

Vice versa if $(\mathfrak{g}, \{,\})$ is a super vector space with a superbracket, we immediately can define its functor of points $\mathbf{g}$. $\mathbf{g}$ is a Lie superalgebra because we have a bracket on $\mathbf{g}(A)$ defined as

$$[a\otimes v, b\otimes w] = (-1)^{p(b)p(v)} ab\otimes \{v,w\}$$

The previous calculation worked backwards proves that $[,]$ is Lie bracket.

With an abuse of language we will call Lie superalgebra both the supervector space $\mathfrak{g}$ with a superbracket $[,]$ and the functor $\mathbf{g}$ as defined in 2.5.

**Observation 2.8.** In [7] is given the notion of a Lie super algebra associated to an affine supergroup. In this work we show that the Lie superalgebra associated to $\mathbf{GL}(m|n)$ is $\mathbf{End}(k^{m|n})$. We will denote $\mathbf{End}(k^{m|n})$ with $\mathfrak{gl}(m|n)$ as supervector space and with $\mathbf{gl}(m|n)$ as its functor of points. The purpose of this paper does not allow for a full description of such correspondence, all the details and the proofs can be found in [7].

# 3 Summary and observations on results by Berele and Regev

In this section we want to review some of the results in [3, 6]. We wish to describe the correspondence between tensorial representations of the superalgebra $\mathfrak{gl}(m|n)$ and representations of the permutation group. This correspondence is obtained using the double centralizer theorem. (Note: in [3] $\mathfrak{gl}(m|n)$ is denoted by $\mathfrak{pl}$).

Let $V = k^{m|n}$ and let $T(V) = \bigoplus_{r\geq 0} T^r(V)$ be the tensor superspace (see Example 2.2 (2)).

We want to define on $T^r(V)$ two actions: one by $S_r$ the permutation group and the other by the Lie superalgebra $\mathfrak{gl}(m|n)$.

Let $\sigma = (i,j) \in S_r$ and let $\{v_i\}_{1\leq i\leq m+n}$ be a basis of $V$ ($v_1 \dots v_m$ even elements and $v_{m+1} \dots v_{m+n}$ odd ones. Let's define:

$$(v_1 \otimes \cdots \otimes v_r) \cdot \sigma =_{\text{def}} \epsilon v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}$$

where $\epsilon = -1$ when $v_i$ and $v_j$ are both odd and $\epsilon = 1$ otherwise. This defines a representation $\tau_r$ of $S_r$ in $T^r(V)$. The proof of this fact can be found in [3] pg 122-123.

Consider now the action $\theta_r$ of the Lie superalgebra $\mathfrak{gl}(m|n)$ on $T^r(V)$ given by derivations:

$$\theta_n(X)(v_1 \otimes \cdots \otimes v_r) =_{\text{def}} \textstyle\sum_i (-1)^{s(X,i)} v_1 \otimes X(v_i) \otimes \cdots \otimes v_r$$

$$g \in \mathfrak{gl}(V)(A), \quad v_i \in V(A), \quad A \in (\text{salg})$$

with $s(X,i) = p(X)o(i)$ where $o(i)$ denotes the number of odd elements among $v_1 \dots v_i$.

One can see that this is a Lie superalgebra action i.e. it preserves the superbracket and that it extends to an action $\theta$ of $\mathfrak{gl}(m|n)$ on $T(V)$ (this is proved in [3] 4.7).

In [3] Theorem 4.14 and Remark 4.15 is proved the important double centralizer theorem:

**Theorem 3.1.** *The algebras $\tau(S_r)$ and $\theta(\mathfrak{gl}(m|n))$ are each the centralizer of the other in* $\text{End}(T^r(V))$.

This result establishes a one to one correspondence between irreducible tensor representations of $S_r$ occurring in $\tau_r$ and those of $\mathfrak{gl}(m|n)$ occurring in $\theta_r$.

These representations are parametrized by partitions $\lambda$ of the integer $r$. In [3] §3 and §4 is worked out completely the structure of irreducible tensor representations of $\mathfrak{gl}(m|n)$ arising in this way. We are interested in their dimensions.

**Definition 3.2.** Let $t_1 < \cdots < t_m < u_1 < \cdots < u_n$ be integers and $\lambda$ a partition of $r$ corresponding to a diagram $D_\lambda$. A filling $T_\lambda$ of $D_\lambda$ is a *semistandard or superstandard tableau* if

1. The part of $T_\lambda$ filled with the $t$'s is a tableaux.
2. The $t$'s are non decreasing in rows and strictly increasing in columns.
3. The $u$'s are non decreasing in columns and strictly increasing in rows.

As an example that will turn out to be important later let's look at $m = n = 1$, $t_1 = 1$, $u_1 = 2$ and $r = 2$. We can have only two partitions: $\lambda = (2)$, $\lambda = (1,1)$. Each partition admits two fillings:

$$\begin{array}{ccc} \lambda = (2) & 1\ \ 1 & 1\ \ 2 \\ \\ \lambda = (1,1) & \begin{matrix} 1 \\ 2 \end{matrix} & \begin{matrix} 2 \\ 2 \end{matrix} \end{array}$$

By Theorem 3.17, 3.18 and 4.17 in [3] we have the following:

**Theorem 3.3.** *The irreducible representations of $\mathfrak{gl}(m|n)$ occurring in $\theta_r$ are parametrized by partitions of $\lambda$ of the integer $r$. The irreducible representations associated to the shape $\lambda$ has dimension equal to the number of semistandard tableaux of shape $\lambda$.*

**Observation 3.4.** This theorem tells us immediately that we have no one dimensional representations of $\mathfrak{gl}(m|n)$ occurring in $\theta_r$, if $n > 0$. In fact one can generalize the Example 3.2 to show that since the odd variables allow repetitions on rows, we always have more than one filling for each shape. However there exists a polynomial representation of $\mathfrak{gl}(m|n)$ of dimension one, namely the supertrace ([1] pg. 100):

$$\begin{array}{rcl} \mathfrak{gl}(m|n) & \longrightarrow & k \cong \mathrm{End}(k) \\ A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} & \mapsto & str(A) =_{\mathrm{def}} tr(X) - tr(W) \end{array}$$

This shows that the tensor representations described in [3] do not exhaust all polynomial representations of $\mathfrak{gl}(m|n)$, for $n > 0$.

# 4 Tensor representations of the general linear supergroup

Let's start by introducing the notion of supergroup and of Lie super algebra representation from a functorial point of view.

**Definition 4.1.** Given an affine algebraic supergroup $\mathbf{G}$ we say that $\mathbf{G}$ acts on a super vector space $W$, if we have a natural transformation:

$$r : \mathbf{G} \longrightarrow \mathbf{End}(W)$$

In other words, if we have for a fixed $A \in$ (salg) a functorial morphism $r_A : \mathbf{G}(A) \longrightarrow \mathbf{End}(W)(A)$. If $W \cong k^{m|n}$ we can identify $r_A(g)$ with a matrix in $\mathbf{End}(W)(A)$ (see Example 2.2 (3)).

Let $V$ be a finite dimensional super vector space. Define:

$$\rho_r : \mathbf{GL}(V) \longrightarrow \mathbf{End}(T^r(V))$$

$$\rho_{r,A}(g) v_1 \otimes \cdots \otimes v_n =_{\text{def}} g(v_1) \otimes \cdots \otimes g(v_n),$$

$$g \in \mathbf{GL}(V)(A), \quad v_i \in \mathbf{V}(A), \quad A \in \text{(salg)}$$

This is an action of $\mathbf{GL}(V)$ on $T^r(V)$, that can be easily extended to the whole $T(V)$.

We now introduce the concept of a Lie superalgebra representation using the functorial language.

**Definition 4.2.** Given a Lie superalgebra $\mathbf{g}$ we say that $\mathbf{g}$ acts on a super vector space $W$, if we have a natural transformation:

$$t : \mathbf{g} \longrightarrow \mathbf{End}(W)$$

preserving the Lie bracket, that is for a fixed $A \in$ (salg), we have a Lie algebra morphism $t_A : \mathbf{g}(A) \longrightarrow \mathrm{End}(W)(A)$. It is easy to verify that this is equivalent to ask that we have a morphism of Lie superalgebras:

$$T : \mathfrak{g} \longrightarrow \mathrm{End}(W)$$

i.e. a super vector space morphism preserving the superbracket. This agrees with the definition of Lie superalgebra representation in [3], which we also recalled in §3.

We are interested in the action $\theta_r$ of $\mathfrak{gl}(V)$, the Lie superalgebra of $\mathbf{GL}(V)$ on $T^r(V)$ introduced in Section 3.

Let's assume from now on $V = k^{m|n}$. Denote with $\{\mathrm{e}_{ij}\}$ the graded canonical basis for the supervector space $\mathfrak{gl}(m|n)$, with $p(\mathrm{e}_{ij}) = p(i) + p(j)$.

**Definition 4.3.** Consider the following functor $\mathbf{E}_{ij}$ : (salg)$\longrightarrow$(sets), $1 \leq i \neq j \leq m+n$:

$$\mathbf{E}_{ij}(A) = \{I + x\mathrm{e}_{ij} | x \in A_k, k = p(\mathrm{e}_{ij})\}$$

This is an affine supergroup functor represented by $k^{1|0}$ if $p(i)+p(j)$ is even, by $k^{0|1}$ if it is odd. We call $\mathbf{E}_{ij}$ a *one parameter subgroup functor.*

Consider also the functor $\mathbf{H}_i$ : (salg)$\longrightarrow$(sets):

$$\mathbf{H}_i(A) = \{I + (x-1)\mathrm{e}_{ii} | x \in A_0 \setminus \{0\}\}$$

This is also an affine supergroup functor represented by $(k^{1|0})^\times$, the multiplicative group of the ground field $k$.

**Theorem 4.4.** *1. The affine supergroup functor* $GL(m|n)$ *is generated by the subgroup functors* $\{\mathbf{E}_{ij}, \mathbf{H}_i\}$*, that is the group* $GL(m|n)(A)$ *is generated by* $\{\mathbf{E}_{ij}(A), \mathbf{H}_i(A)\}$ *for all* $A \in$ (salg).

*2. The Lie superalgebra* $\mathbf{gl}(m|n)$ *is generated by the functors* $\mathbf{e}_{ij}$ *where:*

$$\mathbf{e}_{ij} = \{a \otimes e_{ij} | p(a) = p(i) + p(j)\}$$

**Proof**. (2) is immediate. For (1) it is enough to prove $\mathbf{E}_{ij}(A)$ generate the following (see [17] pg. 117):

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \qquad \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$$

The fact they generate the first type of matrices comes from the classical theory. The fact they generate the other types is immediate. $\square$

Let $\mathrm{GL}(m)$ and $\mathrm{GL}(n)$ denote the general linear group of the ordinary vector spaces $V_0 = k^m$ and $V_1 = k^n$. Consider now the action of the (non super) group $\mathbf{GL}(m) \times \mathbf{GL}(n)$ on the ordinary vector space $V = V_0 \oplus V_1$ and also the action of its Lie algebra $\mathfrak{gl}(m) \times \mathfrak{gl}(n)$ on the same space. We can build the diagonal action $\rho^0$ of $\mathbf{GL}(m) \times \mathbf{GL}(n)$ on the space of tensors $T(V)$ (again $V$ is viewed disregarding the grading) and also the usual action $\theta^0$ by derivation of $\mathfrak{gl}(m) \times \mathfrak{gl}(n)$ on the same space.

**Lemma 4.5.**

$$< \rho^0(GL(m) \times GL(n)) >=< \theta^0(\mathfrak{gl}(m) \times \mathfrak{gl}(n)) >$$

*where* $< S >$ *denotes the subalgebra generated by the set* $S$ *inside* $\mathrm{End}(V)$ *the endomorphism of the ordinary vector space* $V = k^{m+n}$.

**Proof**. This is a consequence of a classical result, see for example [10] 8.2. □

**Theorem 4.6.**

$$< \rho_{r,A}(\mathbf{GL}(m|n)(A)) >_A=< \theta_{r,A}(\mathfrak{gl}(m|n)(A)) >_A \qquad A \in (\mathrm{salg}).$$

*where* $< S >_A$ *denotes the subalgebra generated by the set* $S$ *inside* $\mathbf{End}(V)(A)$.

**Proof**. Since $\mathbf{GL}(m|n)$ is generated by $\{\mathbf{E}_{ij}, \mathbf{H}_i\}$ and $\mathfrak{gl}(m|n)$ is generated by $\{\mathbf{e}_{ij}\}$ it is enough to show that

$$\rho_{r,A}(\mathbf{E}_{ij}(A)) \in \theta_{r,A}(\mathbf{gl}(V)(A)), \qquad \theta_{r,A}(\mathbf{e}_{ij}(A)) \in \rho_{r,A}(\mathbf{GL}(V)(A))$$

When $p(i) + p(j)$ is even this is an easy consequence of Lemma 4.5.

Now the case when $p(i) + p(j)$ is odd. Let $D_{ij}$ be the derivation corresponding to the elementary matrix $\mathrm{e}_{ij}$. So we have that $\mathbf{e}_{ij}(A) = \alpha_1 \otimes D_{ij}$, $\alpha \in A_1$. We claim that

$$\theta_{r,A}(\mathbf{e}_{ij}(A)) = 1(A) - \rho_{r,A}(\mathbf{E}_{ij})(A)$$

This is a calculation. □

**Corollary 4.7.** *There is a one to one correspondence between the irreducible representations of* $S_r$ *and the irreducible representations of* $\mathbf{GL}(m|n)$ *occurring in* $\rho_r$.

**Observation 4.8.** By Corollary 4.7 and Theorem 3.3 we have that also the irreducible representations occuring in $\rho_r$ of $\mathbf{GL}(m|n)$ are parametrized by partitions of the integer $r$. However by Observation 3.4 we have that there

is no one dimensional irreducible representation hence also for $GL(m|n)$ we miss an important representation, namely the Berezinian:

$$\begin{array}{ccc} \mathbf{GL}(m|n)(A) & \longrightarrow & A \cong \mathbf{End}(k)(A) \\ \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} & \mapsto & det(W)^{-1}det(X - YW^{-1}Z) \end{array}$$

This shows that the tensor representations of $\mathbf{GL}(m|n)$ so not exhaust all polynomial representations of $\mathbf{GL}(m|n)$, for $n > 0$.

The Berezinian representation has been described by Deligne and Morgan in [5] pg 60, in a natural way as an action of $\mathbf{GL}(V)$ on Ext group $\mathrm{Ext}^m_{\mathrm{Sym}^*(V^*)}(A, \mathrm{Sym}^*(V^*))$. Ext plays the same role as the antisymmetric tensors in this super setting. It would be interesting to see if there are other objects that give raise to representations which are not among those previously described.

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